Interface fluctuations under shear

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Coarsening systems under uniform shear display a long time regime characterized by the presence of highly stretched and thin domains. The question then arises whether thermal fluctuations may actually destroy this layered structure. To address this problem in the case of nonconserved dynamics, we study an anisotropic version of the Burgers equation, constructed to describe thermal fluctuations of an interface in the presence of a uniform shear flow. As a result, we find that stretched domains are only marginally stable against thermal fluctuations in $d=2$, whereas they are stable in $d=3$.

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The dynamics of phase separation in systems quenched below their critical temperature is substantially modified when an external shear is applied. This is true both for the case of conserved dynamics (spinodal decomposition in binary fluids) and nonconserved dynamics (coarsening in Ising spin systems). In the simplest theoretical case in which the sample is confined between two boundaries moving at a constant relative velocity, intuition suggests that the growing domain structure becomes anisotropic, with domains highly stretched in the direction of the flow. This fact is now well established by many experimental [1] and theoretical investigations [2–6]: it has been shown that the growth of the domains is enhanced along the flow direction, while it is less clear whether the transverse growth rate is unaffected by the shear or depressed by it. In either case, the long time effect of the shear is to form a structure of very long and (relatively) thin domains.

A natural question in this context is to what extent this layered structure is stable against thermal fluctuations: long and thin domains may develop fluctuations, transverse to their main axis, that grow large enough to break them. Similar stretching-and-breaking mechanisms have been proposed before [5,6], in particular to argue that the anisotropic domain growth may eventually reach a steady state at late times [5]. However, the effect of the shear is not only to stretch the domains, but also to smooth their surfaces, and the net result of the two effects is difficult to predict. In order to clarify this problem, a suitable model for the growth of a surface under shear must be analyzed.

Unfortunately, not many analytic studies of domain growth under shear exist. In particular, conserved dynamics and spinodal decomposition have been analytically considered mainly in the limit of infinite dimension of the order parameter [3], where no domain interfaces are present. The situation is better in the simpler case of nonconserved dynamics, where zero-temperature coarsening under shear has recently been studied analytically [4]. Our task is therefore to study the effect of thermal fluctuations in the case of nonconserved dynamics under shear. To this end, we will introduce a stochastic equation for a scalar field $h(\vec{x},t)$ representing the height of a fluctuating interface above its flat ground state. This equation will take into account the smoothing effect of the shear flow, which in fact gives origin to a nonlinearity of the Burgers type.

Consider an interface separating two regions with opposite order parameter. When the temperature is small but nonzero, there will be some fluctuations, resulting in a nonflat profile parametrized by the height $h(\vec{x},t)$. In the following, we will indicate with d the total dimension of the space, and with d' the dimension of the substrate spanned by $\vec{x}$, that is, $d=d'+1$. In the absence of shear, the free-energy cost of a nonflat profile is given by $F=(\sigma/2)\int d\vec{x}(\vec{\nabla}h)^2$, where σ is the surface tension, and the corresponding Langevin equation coincides with the standard Edwards-Wilkinson (EW) growth equation [7],

$$\partial_t h=\nu\nabla^2 h+\xi, \tag{1}$$

where ν is a diffusion coefficient (equal to σ divided by a kinetic coefficient) and $\xi(\vec{x},t)$ is a δ-correlated noise,

$$\langle\xi(\vec{x},t)\xi(\vec{x}',t')\rangle=D\,\delta(\vec{x}-\vec{x}')\,\delta(t-t'). \tag{2}$$

The noise strength D is proportional to the temperature. As is well known, due to its linearity a simple scaling analysis of the EW equation gives the exact critical exponents. Under the rescaling $x\to bx$, $t\to b^{z_0}t$, and $h\to b^{\chi_0}h$, we have $\nu\to b^{z_0-2}\nu$ and $D\to b^{z_0-2\chi_0-d+1}D$. By imposing scale invariance, we obtain $z_0=2$ and $\chi_0=(3-d)/2$. Even though the EW equation is not suitable for describing interface fluctuations when a shear is present, it is interesting to see what the EW exponents would predict for the stability of the stretched domains. For $d=2$, the investigation of [4] gives two length scales, $L_\parallel(t)=O(t)$ and $L_\perp=O(1)$ (up to logarithmic factors). Within the context of the EW equation, the transverse fluctuations will grow as

$$h\sim t^{\chi_0/z_0}F(t/L_\parallel^{z_0}), \tag{3}$$

where F is a scaling function with the limiting forms $F(0)=\text{const}$, $F(s)\sim s^{-\chi_0/z_0}$ for $s\to\infty$. In writing down (3), we have assumed that $L_\parallel$ is a fixed length scale, while actually it is growing with time. The interpretation of (3) is, however, simple. If $t\ll L_\parallel^{z_0}$, the domains are coarsening faster than the interfacial fluctuations, and we can effectively set $L_\parallel$ to infinity. Then $h\sim t^{\chi_0/z_0}$. On the other hand, if $t\gg L_\parallel^{z_0}$, the fluctuations are coarsening faster than the domains and eventually equilibrate on the scale $L_\parallel$. This corresponds to the large argument limit of the scaling variable in (3), giving $h \sim L_\parallel^{\chi_0}$. Combining these two limits gives

$$h \sim \min(t^{\chi_0/z_0}, L_\parallel^{\chi_0}). \tag{4}$$

In $d=2$, this gives $h \sim t^{1/4}$, while $L_\perp = 0(1)$. Thus, thermal fluctuations would eventually disrupt the domain structure and destroy the coarsening state. In $d=3$, on the other hand, it has been found that [4] $L_\parallel(t) = O(t^{3/2})$ and $L_\perp(t) = O(t^{1/2})$. In this case, $\chi_0 = 0$ (logs), so conventional EW thermal roughening would not destroy the domains, since $h \ll L_\perp$. As we shall see, the inclusion of shear in Eq. (1) will modify these results. In particular, the instability of the domains in $d=2$ will be reduced, while stability in $d=3$ will be confirmed.

When a shear flow is present, the EW equation must be modified. We will consider a standard shear velocity profile $\vec{u}$, with flow along the x direction and shear gradient perpendicular to the surface. Let us label this last direction by z, such that

$$\vec{u} = \gamma z \vec{e}_x, \tag{5}$$

where γ is the shear rate. In this way, we break the symmetry between the x direction and the remaining $(d'-1)$ directions $\vec{x}_\perp$. Note that the growth is orthogonal to the shear flow: this is what happens to the domain walls in the long-time limit of a coarsening process under shear. The field $h(\vec{x},t)$ is now dragged in the x direction by an amount proportional to h itself, that is, $\partial_t \to \partial_t + \gamma h \partial_x$. The correct equation for $h(\vec{x},t)$ therefore becomes

$$\partial_t h + \gamma h \partial_x h = \nu_x \partial_{xx} h + \nu_\perp \nabla_\perp^2 h + \xi, \tag{6}$$

where we have introduced separate diffusion constants, ν_x and $\nu_\perp$. For $d'=1$ $(d=2)$, Eq. (6) is nothing other than the Burgers equation [8]. This can be mapped onto the Kardar-Parisi-Zhang (KPZ) equation [9] with space-correlated noise, via the transformation $h = \partial_x \hat{h}$, i.e., $h\partial h \to \frac{1}{2}\partial(\partial\hat{h})^2$, yielding the KPZ equation for $\hat{h}$. Such a case has been studied in [10]. In the generic dimension, Eq. (6) is anisotropic and was first introduced in [11], in the context of a model of sandpiles, and further studied in [12]. A standard method for the analysis of stochastic nonlinear equations is the dynamic renormalization-group (RG) approach, first used in this context in [13]. Here, we will briefly review those RG results for Eq. (6) that are most relevant for our purpose. We start the analysis of Eq. (6) by finding the bare scaling dimensions of the parameters. Under the anisotropic rescaling,

$$x \to bx, \quad \vec{x}_\perp \to b^\zeta \vec{x}_\perp, \quad t \to b^z t, \quad h \to b^\chi h, \tag{7}$$

we obtain

$$\nu_x \to b^{z-2}\nu_x, \quad \nu_\perp \to b^{z-2\zeta}\nu_\perp, \quad \gamma \to b^{\chi+z-1}\gamma,$$
$$D \to b^{z-2\chi-1-(d-2)\zeta} D. \tag{8}$$

Inserting the EW exponents χ_0 and z_0 found above for $\gamma = 0$ into the scaling equation for γ gives $\gamma \to b^{(5-d)/2}\gamma$, showing that $d_c = 5$ is the critical dimension below which the nonlinearity becomes relevant. Thus, for $d<5$ the EW exponents are no longer correct and an RG approach becomes necessary.

The Burgers equation is known to be invariant under a Galilean transformation. In the present context, we have an equivalent symmetry, namely the invariance of Eq. (6) under a coordinate transformation that preserves the original form of the shear flow. Indeed, from Eq. (5) we have $x \sim \gamma h t$, and therefore if we vertically shift the interface, $h \to h + h_0$, the equation is invariant provided that we make the transformation $x \to x + \gamma h_0 t$. Basically, this is just translational invariance in the z direction. This exact symmetry must be preserved by the RG transformation. An immediate consequence is that γ cannot be perturbatively corrected in a RG analysis, and we must set to zero its bare scaling dimension in Eq. (8). In this way, a relation between dynamic and growth exponents is obtained:

$$\chi + z = 1. \tag{9}$$

The remarkable feature of Eq. (6) is that there are two further parameters whose bare scaling dimensions are not perturbatively changed, namely $\nu_\perp$ and the noise strength D. To see this, we introduce the response function, defined in Fourier space as $G(\vec{k},\omega) = \langle \partial h(\vec{k},\omega)/\partial \xi(\vec{k},\omega)\rangle$. The bare response function, $G_0(\vec{k},\omega)$, is obtained from Eq. (6) with $\gamma=0$: $G_0(\vec{k},\omega) = [-i\omega + \nu_x k_x^2 + \nu_\perp \vec{k}_\perp^2]^{-1}$. The exact response function has the form $G(\vec{k},\omega) = [-i\omega + \nu_x k_x^2 + \nu_\perp \vec{k}_\perp^2 - \Sigma(\vec{k},\omega)]^{-1}$. Here $\Sigma(\vec{k},\omega)$ is the self-energy function, which can be calculated perturbatively in γ within an RG scheme. The key point is that the γ vertex in Eq. (6) carries in Fourier space a factor k_x. This means that the self-energy, and also the perturbatively corrected noise correlator, carry factors of k_x at every order in perturbation theory (in fact, their leading corrections are of order k_x^2). It follows that there can be no terms of order $O(k_\perp^2)$ in the self-energy, contributing to a renormalization of $\nu_\perp$, nor $O(1)$ terms in the renormalized noise, contributing to the renormalization of D [11]. We can therefore set to zero, in Eq. (8), the bare scaling dimensions of $\nu_\perp$ and D, obtaining in this way two extra relations among the fixed point exponents:

$$z = 2\zeta, \tag{10}$$

$$z = 2\chi + 1 + (d-2)\zeta. \tag{11}$$

From Eqs. (9)–(11), we obtain, for $d \leqslant 5$ [11]

$$z = \frac{6}{8-d}, \quad \chi = \frac{2-d}{8-d}, \quad \zeta = \frac{3}{8-d}. \tag{12}$$

From Eqs. (7) and (12), we can infer the scaling form for the correlation function in Fourier space, defined by $\langle h(\vec{k},\omega) h(\vec{k}',\omega')\rangle = C(\vec{k},\omega)\,\delta(\vec{k}+\vec{k}')\,\delta(\omega+\omega')$. We have

$$C(\vec{k},\omega)=\frac{1}{k_x^{2z}}f\left(\frac{\omega}{k_x^z},\frac{|\vec{k}_\perp|}{k_x^\zeta}\right), \tag{13}$$

where f is a scaling function.

It is useful to rescale coordinates and field in order to identify the effective coupling constant of Eq. (6). The change of variables,

$$h\rightarrow(D/\nu_x^{2-d/2}\nu_\perp^{d/2-1})^{1/2}h,\quad x_\perp\rightarrow(\nu_\perp/\nu_x)^{1/2}x_\perp,$$
$$t\rightarrow(1/\nu_x)t,\quad \xi\rightarrow(D\nu_x^{d/2}/\nu_\perp^{d/2-1})^{1/2}\xi, \tag{14}$$

amounts to setting $\nu_x=\nu_\perp=D=1$ and replacing the vertex γ by the effective vertex

$$\hat{\gamma}=\frac{\gamma D^{1/2}}{\nu_x^{2-d/4}\nu_\perp^{d/4-1/2}}. \tag{15}$$

As expected, for $d=d_c=5$ this quantity is dimensionless. Using standard RG methods (see, for example, [13,11,10]) it is possible to obtain the one-loop RG flow equation for the effective coupling constant $U=c\hat{\gamma}^2$, where $c=\Gamma(3-d/2)/[8(4\pi)^{d/2-1}]$. The flow equation reads

$$\frac{dU}{dl}=(5-d)U-\tfrac{1}{2}(8-d)^2U^2. \tag{16}$$

The trivial fixed point, $U=0$, becomes stable above the critical dimension $d_c=5$, giving the EW exponents z_0,χ_0, corresponding to $\zeta=\zeta_0=1$. No other physical ($U>0$) fixed point exists in this phase. The effect of the shear on thermal roughening is therefore negligible above dimension $d_c=5$ and domain-wall fluctuations are isotropic (in the d'-dimensional subspace parallel to the mean orientation of the wall). On the other hand, in dimension $d<5$ there is an attractive fixed point at $U=2\epsilon/9+O(\epsilon^2)$, with $\epsilon=d_c-d$. The effective coupling constant is of order ϵ, such that close to the critical dimension the RG expansion is under control. This fixed point controls the large-scale behavior of Eq. (6) for $d<5$ and is associated with the nontrivial exponents (12).

The scaling exponent χ of the interface height is zero for $d=2$, while it is negative, $\chi=-\frac{1}{5}$, for $d=3$. From Eq. (4), with χ_0 and z_0 replaced by the new exponents χ and z, we see that in the two-dimensional case the highly elongated domains are only marginally stable against thermal fluctuations, since $L_\parallel^\chi=O(1)$, $t^{\chi/z}=O(1)$, and $L_\perp=O(1)$ (up to logarithms in each case). A thermally induced stretching-and-breaking mechanism cannot, therefore, be excluded in this case. On the other hand, for $d=3$ a negative value of χ means that thermal fluctuations of the interfaces saturate at late times. Thus, in the three-dimensional case thermal roughening is depressed by the shear, and the flow-induced layered structure of the domains is stable against thermal fluctuations.

In this work, we have shown that interface fluctuations under shear, in a system with nonconserved order parameter, can be described by a stochastic differential equation, with an anisotropic Burgers nonlinearity, and that the critical exponents are known exactly. In this way, it was possible to assess the stability of domains in the late time regime of a system subject to a uniform shear flow.

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